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Handling Equality in Monodic Temporal Resolution

Boris Konev[1⋆], Anatoli Degtyarev[2], and Michael Fisher[1]

[1] Department of Computer Science, University of Liverpool, Liverpool L69 7ZF, U.K.
{B.Konev, M.Fisher}@csc.liv.ac.uk
[2] Department of Computer Science, King's College London, Strand, London WC2R 2LS, U.K.
anatoli@dcs.kcl.ac.uk

Abstract. First-order temporal logic is a concise and powerful notation, with many potential applications in both Computer Science and Artificial Intelligence. While the full logic is highly complex, recent work on monodic first-order temporal logics has identified important enumerable and even decidable fragments including the guarded fragment with equality. In this paper, we specialise the monodic resolution method to the guarded monodic fragment with equality and first-order temporal logic over expanding domains. We introduce novel resolution calculi that can be applied to formulae in the normal form associated with the clausal resolution method, and state correctness and completeness results.

1 Introduction

First-order temporal logic (FOTL) is a powerful notation with many applications in formal methods. Unfortunately, this power leads to high complexity, most notably the lack of recursive axiomatisations for general FOTL. Recently, significant work has been carried out in defining *monodic* FOTL, a class of logics retaining finite axiomatisation, with both tableau and resolution systems being under development [12, 3]. However, until now, little work has been carried out concerning monodic FOTL with equality and *no* practical proof technique for such logics has been proposed. In real applications of formal specification, the notion of equality plays a key role and so, in this paper, we extend and adapt our clausal resolution approach, which has already been successfully applied to a variety of monodic logics, to the case of monodic FOTL with equality. In particular, we develop a decision procedure for the guarded monodic fragment of FOTL with equality over constant and expanding domains; decidability of this fragment has been established in [9]. However, decidability was given there using model-theoretic techniques, and practical proof techniques were not considered. In this paper we address the problem of producing a practical proof technique for this class of logic through extension of the clausal resolution method for monodic temporal logics. A complete temporal resolution calculus for the monodic temporal fragment *without equality* for the constant domain case has been presented in [3]. The expanding domain case has been announced in [11] and proved in a technical report [4]. Finally, we also point to a *fine-grained* superposition calculus for the monodic guarded fragment with equality interpreted over expanding domains. This suggests adapting our previous work on fine-grained temporal resolution [11] and combining this with (parts of) the superposition calculus for the (first-order) guarded fragment with equality given in [7].

⋆ On leave from Steklov Institute of Mathematics at St.Petersburg

2 First-Order Temporal Logic

First-Order (discrete linear time) Temporal Logic, FOTL, is an extension of classical first-order logic with operators that deal with a linear and discrete model of time (isomorphic to $\mathbb{N}$, and the most commonly used model of time). The first-order function-free temporal language is constructed in a standard way [6, 10] from: *predicate symbols* $P_0, P_1, \ldots$, each of which is of some fixed arity (null-ary predicate symbols are called *propositions*); *equality*, denoted by the symbol $\approx$[3]; *individual variables* $x_0, x_1, \ldots$; *individual constants* $c_0, c_1, \ldots$; *Boolean operators* $\wedge$, $\neg$, $\vee$, $\Rightarrow$, $\equiv$, **true** ('true'), **false** ('false'); *quantifiers* $\forall$ and $\exists$; together with *unary temporal operators*, such as[4] $\square$ ('always in the future'), $\diamondsuit$ ('sometime in the future'), and $\bigcirc$ ('at the next moment').

Formulae in FOTL are interpreted in *first-order temporal structures* of the form $\mathfrak{M} = \langle D_n, I_n \rangle$, $n \in \mathbb{N}$, where every D_n is a non-empty set such that whenever $n < m$, $D_n \subseteq D_m$, and I_n is an interpretation of predicate and constant symbols over D_n. We require that the interpretation of constants is *rigid*. Thus, for every constant c and all moments of time $i, j \geq 0$, we have $I_i(c) = I_j(c)$. The interpretation of $\approx$ is fixed as the identity on every D_n. The interpretation of predicate symbols is flexible. A *(variable) assignment* $\mathfrak{a}$ is a function from the set of individual variables to $\cup_{n \in \mathbb{N}} D_n$. (This definition implies that variable assignments are also rigid.) We denote the set of all assignments by $\mathfrak{V}$.

For every moment of time n, there is a corresponding *first-order* structure, $\mathfrak{M}_n = \langle D_n, I_n \rangle$; the corresponding set of variable assignments $\mathfrak{V}_n$ is a subset of the set of all assignments, $\mathfrak{V}_n = \{\mathfrak{a} \in \mathfrak{V} \mid \mathfrak{a}(x) \in D_n$ for every variable $x\}$; clearly, $\mathfrak{V}_n \subseteq \mathfrak{V}_m$ if $n < m$. Intuitively, FOTL formulae are interpreted in sequences of *worlds*, $\mathfrak{M}_0, \mathfrak{M}_1, \ldots$ with truth values in different worlds being connected via temporal operators.

The *truth* relation $\mathfrak{M}_n \models^{\mathfrak{a}} \phi$ in a structure $\mathfrak{M}$, *only for those assignments* $\mathfrak{a}$ *that satisfy the condition* $\mathfrak{a} \in \mathfrak{V}_n$, is defined inductively in the usual way under the following understanding of temporal operators:

$$
\begin{array}{l}
\mathfrak{M}_n \models^{\mathfrak{a}} \bigcirc\phi \text{ iff } \mathfrak{M}_{n+1} \models^{\mathfrak{a}} \phi; \\
\mathfrak{M}_n \models^{\mathfrak{a}} \diamondsuit\phi \text{ iff there exists } m \geq n \text{ such that } \mathfrak{M}_m \models^{\mathfrak{a}} \phi; \\
\mathfrak{M}_n \models^{\mathfrak{a}} \square\phi \text{ iff for all } m \geq n, \mathfrak{M}_m \models^{\mathfrak{a}} \phi.
\end{array}
$$

$\mathfrak{M}$ is a *model* for a formula ϕ (or ϕ is *true* in $\mathfrak{M}$) if there exists an assignment $\mathfrak{a}$ such that $\mathfrak{M}_0 \models^{\mathfrak{a}} \phi$. A formula is *satisfiable* if it has a model. A formula is *valid* if it is true in any temporal structure under any assignment.

The models introduced above are known as *models with expanding domains*. Another important class of models consists of *models with constant domains* in which the class of first-order temporal structures, where FOTL formulae are interpreted, is restricted to structures $\mathfrak{M} = \langle D_n, I_n \rangle$, $n \in \mathbb{N}$, such that $D_i = D_j$ for all $i, j \in \mathbb{N}$. The notions of truth and validity are defined similarly to the expanding domain case. It is known [14] that satisfiability over expanding domains can be reduced to satisfiability over constant domains.

[3] We are using the symbol $\approx$ for equality in the object language in order to avoid confusion with the symbol $=$ for equality in the meta language.

[4] W.r.t. satisfiability, the binary temporal operators U ('until') and W ('weak until') can be represented using the unary temporal operators [6, 2] with a linear growth in the size of a formula.

Example 1. The formula $\forall x P(x) \wedge \Box(\forall x P(x) \Rightarrow \bigcirc \forall x P(x)) \wedge \Diamond \exists x \neg P(x)$ is unsatisfiable over both expanding and constant domains; the formula $\forall x P(x) \wedge \Box(\forall x (P(x) \Rightarrow \bigcirc P(x))) \wedge \Diamond \exists x \neg P(x)$ is unsatisfiable over constant domains but has a model with an expanding domain.

This logic is complex. It is known that even "small" fragments of FOTL, such as the *two-variable monadic* fragment (all predicates are unary), are not recursively enumerable [13, 10]. However, the set of valid *monodic* formulae *without equality* is known to be finitely axiomatisable [15].

Definition 1 (Monodic fragment). *An* FOTL*-formula* ϕ *is called* monodic *if any subformulae of the form* $\mathcal{T}\psi$*, where* $\mathcal{T}$ *is one of* $\bigcirc$*,* $\Box$*,* $\Diamond$ *(or* $\psi_1 \mathcal{T} \psi_2$*, where* $\mathcal{T}$ *is one of* U*,* W*), contains at most one free variable.*

The addition of either equality or function symbols to the monodic fragment leads to the loss of recursive enumerability [15]. Moreover, it was proved in [5] that the *two variable monadic monodic fragment with equality* is not recursively enumerable. However, in [9] it was shown that the guarded monodic fragment with equality is decidable[5].

Definition 2 (Guarded monodic fragment with equality). *The formulae of the* guarded monodic fragment MGF *are inductively defined as follows:*

1. *If A is an atom (which can be non-equational, of the form* $P(t_1, \ldots, t_n)$ *an equation, of the form* $s \approx t$*, as well as a logical constant,* **true** *or* **false***), then A is in* MGF*, where* $t_1, \ldots, t_n, s, t$ *are constants or variables.*
2. MGF *is closed under boolean combinations.*
3. *If* $\phi \in$ MGF *and G is an atom (possibly equation), for which every free variable of* ϕ *is among the arguments of G, then* $\forall \overline{x}(G \Rightarrow \phi) \in$ MGF *and* $\exists \overline{x}(G \wedge \phi) \in$ MGF*, for every sequence* $\overline{x}$ *of variables. The atom G is called a* guard.
4. *If* $\phi(x) \in$ MGF *and* $\phi(x)$ *contains at most one free variable, then* $\bigcirc\phi(x) \in$ MGF*,* $\Box\phi(x) \in$ MGF*, and* $\Diamond\phi(x) \in$ MGF.
5. *If* $\phi(x) \in$ MGF *and* $\phi(x)$ *contains exactly one free variable x, then* $\forall x \phi(x)$ *and* $\exists x \phi(x)$ *are in* MGF.

Note 1. Although the standard definition of the guarded fragment (see, for example, [7]) does not contain item 5, its addition does not extend the notion of the guarded fragment: we can always choose $x \approx x$ as the guard for $\forall x$ and $\exists x$.

3 Divided Separated Normal Form (DSNF)

Definition 3. *A* temporal step clause *is a formula either of the form* $l \Rightarrow \bigcirc m$*, where l and m are propositional literals, or* $(L(x) \Rightarrow \bigcirc M(x))$*, where* $L(x)$ *and* $M(x)$ *are unary literals. We call a clause of the the first type an (original)* ground *step clause, and of the second type an (original)* non-ground *step clause.*

[5] All cases considered in [5] included formulae of the form $\Box\, \forall x \forall y\, ((P(x) \wedge P(y)) \supset x \approx y)$ or similar non-ground formulae.

Definition 4. *A* monodic temporal problem in Divided Separated Normal Form (DSNF) *is a quadruple* $\langle \mathscr{U}, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$, *where*

1. *the universal part,* $\mathscr{U}$, *is a finite set of arbitrary closed first-order formulae;*
2. *the initial part,* $\mathscr{I}$, *is, again, a finite set of arbitrary closed first-order formulae;*
3. *the step part,* $\mathscr{S}$, *is a finite set of original (ground and non-ground) temporal step clauses; and*
4. *the eventuality part,* $\mathscr{E}$, *is a finite set of eventuality clauses of the form* $\Diamond L(x)$ *(a* non-ground *eventuality clause) and* $\Diamond l$ *(a* ground eventuality *clause), where* l *is a propositional literal and* $L(x)$ *is a unary non-ground literal.*

For a temporal problem, P, $\mathrm{const}(\mathsf{P})$ denotes the set of constants occurring in P.

Note that, in a monodic temporal problem, we disallow two different temporal step clauses with the same left-hand sides. We also disallow occurrences of equality in the step and eventuality parts. These requirements can be easily guaranteed by renaming.

In what follows, we will not distinguish between a finite set of formulae $\mathscr{X}$ and the conjunction $\bigwedge \mathscr{X}$ of formulae within the set. With each monodic temporal problem, we associate the formula $\mathscr{I} \land \Box\mathscr{U} \land \Box\forall x\mathscr{S} \land \Box\forall x\mathscr{E}$. Now, when we talk about particular properties of a temporal problem (e.g., satisfiability, validity, logical consequences etc) we mean properties of the associated formula.

Arbitrary monodic FOTL-formulae can be transformed into DSNF in a satisfiability equivalence preserving way using a renaming technique replacing non-atomic subformulae with new propositions and removing all occurrences of the U and W operators [6, 3]. If the given formula is a guarded monodic formula, then all parts of DSNF (and the associated formula) are guarded monodic formulae. In this case, we call the result of the transformation a *guarded monodic problem*.

4 Calculi

In this section we present two resolution calculi, $\mathfrak{I}_c$ and $\mathfrak{I}_e$, for guarded monodic problems (including equality). These calculi are very similar, but $\mathfrak{I}_c$ is complete for problems featuring constant domains, while $\mathfrak{I}_e$ is complete for those involving expanding domains. These resolution calculi are based on those introduced in [3] for problems *without* equality. Thus, the work described in this section extends previous calculi to allow consideration of equality in guarded monodic problems.

We begin with a number of important definitions.

Definition 5 (Equational augmentation). *Let* P *be a temporal problem. Its (equational) augmentation is the set* $\mathrm{aug}_{=}(\mathsf{P})$ *of step clauses. For every constant* $c \in \mathrm{const}(\mathsf{P})$, *the following clauses are in* $\mathrm{aug}_{=}(\mathsf{P})$.

$$(x \approx c) \Rightarrow \bigcirc(x \approx c), \tag{1}$$

$$(x \not\approx c) \Rightarrow \bigcirc(x \not\approx c). \tag{2}$$

Note that clauses originating from such augmentation are the only step clauses that contain equality.

Definition 6 (Derived/E-Derived Step Clauses). *Let P be a monodic temporal problem, and let*

$$L_{i_1}(x) \Rightarrow \bigcirc M_{i_1}(x), \ldots, L_{i_k}(x) \Rightarrow \bigcirc M_{i_k}(x) \tag{3}$$

be a subset of the set of its original non-ground step clauses, or *clauses from* $\mathrm{aug}_{=}(\mathsf{P})$. *Then formulae of the form*

$$\exists x(L_{i_1}(x) \wedge \ldots \wedge L_{i_k}(x)) \Rightarrow \bigcirc \exists x(M_{i_1}(x) \wedge \ldots \wedge M_{i_k}(x)), \tag{4}$$

$$\forall x(L_{i_1}(x) \vee \ldots \vee L_{i_k}(x)) \Rightarrow \bigcirc \forall x(M_{i_1}(x) \vee \ldots \vee M_{i_k}(x)) \tag{5}$$

are called derived *step clauses. Formulae of the form* (4) *are called* e-derived *step clauses.*

Note that formulae of the form (4) are logical consequences of (3) in the *expanding domain* case; while formulae of the form (4) *and* (5) are logical consequences of (3) in the *constant domain* case. As Example 1 shows, (5) is not a logical consequence of (3) in the expanding domain case.

Definition 7 (Merged Derived/E-Derived Step Clauses). *Let $\{\Phi_1 \Rightarrow \bigcirc \Psi_1, \ldots, \Phi_n \Rightarrow \bigcirc \Psi_n\}$ be a set of derived (e-derived) step clauses or original* ground *step clauses. Then $\bigwedge_{i=1}^{n} \Phi_i \Rightarrow \bigcirc \bigwedge_{i=1}^{n} \Psi_i$ is called a* merged derived (merged e-derived) step clause.

Note 2. In [3], where no equality was considered, instead augmenting a problem with clauses of the form (1) and (2), we defined another derived step clause

$$L(c) \Rightarrow \bigcirc M(c), \tag{6}$$

where $c \in \mathrm{const}(\mathsf{P})$. Note that this clause is equivalent to an e-derived step clause

$$\exists x(L(x) \wedge x \approx c) \Rightarrow \bigcirc \exists x(M(x) \wedge x \approx c).$$

Definition 8 (Full Merged/E-Merged Step Clauses). *Let $\mathscr{A} \Rightarrow \bigcirc \mathscr{B}$ be a merged derived (merged e-derived) step clause, $L_1(x) \Rightarrow \bigcirc M_1(x), \ldots, L_k(x) \Rightarrow \bigcirc M_k(x)$ be original step clauses* or *step clauses from* $\mathrm{aug}_{=}(\mathsf{P})$, *and $A(x) \stackrel{\text{def}}{=} \bigwedge_{i=1}^{k} L_i(x)$, $B(x) \stackrel{\text{def}}{=} \bigwedge_{i=1}^{k} M_i(x)$. Then $\forall x(\mathscr{A} \wedge A(x) \Rightarrow \bigcirc(\mathscr{B} \wedge B(x)))$ is called a* full merged step clause *(full e-merged step clause, resp.). In the case $k = 0$, the conjunctions $A(x)$, $B(x)$ are empty, i.e., their truth value is* **true**, *and the merged step clause is just a merged derived step clause.*

We now present two calculi, $\mathfrak{I}_c$ and $\mathfrak{I}_e$, aimed at the constant and expanding domain cases, respectively. The inference rules of these calculi coincide; the only difference is in the merging operation. The calculus $\mathfrak{I}_c$ utilises merged derived and full merged step clauses; whereas $\mathfrak{I}_e$ utilises merged e-derived and full e-merged step clauses.

Inference Rules. In what follows, $\mathscr{A} \Rightarrow \bigcirc \mathscr{B}$ and $\mathscr{A}_i \Rightarrow \bigcirc \mathscr{B}_i$ denote merged derived (e-derived) step clauses, $\forall x(\mathscr{A}(x) \Rightarrow \bigcirc(\mathscr{B}(x)))$ and $\forall x(\mathscr{A}_i(x) \Rightarrow \bigcirc(\mathscr{B}_i(x)))$ denote full merged (e-merged) step clauses, and $\mathscr{U}$ denotes the (current) universal part of the problem. Thus, $\phi \models \psi$ means that ψ is a (first-order) logical consequence of ϕ.

– *Step resolution rule w.r.t.* $\mathscr{U}$: $\dfrac{\mathscr{A} \Rightarrow \bigcirc\mathscr{B}}{\neg\mathscr{A}}$ $(\bigcirc^{\mathscr{U}}_{res})$, where $\mathscr{U} \cup \{\mathscr{B}\} \models \bot$.
– *Initial termination rule w.r.t.* $\mathscr{U}$*:* The contradiction $\bot$ is derived and the derivation is (successfully) terminated if $\mathscr{U} \cup \mathscr{I} \models \bot$.
– *Eventuality resolution rule w.r.t.* $\mathscr{U}$*:*

$$\frac{\forall x(\mathscr{A}_1(x) \Rightarrow \bigcirc(\mathscr{B}_1(x)))\ldots\forall x(\mathscr{A}_n(x) \Rightarrow \bigcirc(\mathscr{B}_n(x))) \qquad \Diamond L(x)}{\forall x \bigwedge_{i=1}^{n} \neg\mathscr{A}_i(x)} \ (\Diamond^{\mathscr{U}}_{res}),$$

where $\Diamond L(x) \in \mathscr{E}$ and $\forall x(\mathscr{A}_i(x) \Rightarrow \bigcirc\mathscr{B}_i(x))$ are full merged (full e-merged) step clauses such that for all $i \in \{1,\ldots,n\}$, the *loop* side conditions $\forall x(\mathscr{U} \wedge \mathscr{B}_i(x) \Rightarrow \neg L(x))$ and $\forall x(\mathscr{U} \wedge \mathscr{B}_i(x) \Rightarrow \bigvee_{j=1}^{n} (\mathscr{A}_j(x)))$ are both valid first-order formulae.
– *Eventuality termination rule w.r.t.* $\mathscr{U}$*:* The contradiction is derived and the derivation is (successfully) terminated if $\mathscr{U} \models \forall x \neg L(x)$, where $\Diamond L(x) \in \mathscr{E}$.
– *Ground eventuality resolution w.r.t.* $\mathscr{U}$ and *Ground eventuality termination w.r.t.* $\mathscr{U}$*:* These rules repeat the eventuality resolution and eventuality termination rules with the only difference that *ground eventualities* and merged *derived step clauses* are used instead of non-ground eventualities and full merged step clauses.

A *derivation* is a sequence of universal parts, $\mathscr{U} = \mathscr{U}_0 \subseteq \mathscr{U}_1 \subseteq \mathscr{U}_2 \subseteq \ldots$, extended little by little by the conclusions of the inference rules. Successful termination means that the given problem is unsatisfiable. The $\mathscr{I}$, $\mathscr{S}$ and $\mathscr{E}$ parts of the temporal problem are not changed in a derivation.

Example 2. Let us consider an unsatisfiable (over both constant and expanding domains) temporal problem given by

$$\mathscr{I} = \left\{\, i1.\ \ \exists x(x \not\approx c) \,\right\},\ \mathscr{U} = \left\{ \begin{array}{l} u1.\ \ \exists x(P(x)), \\ u2.\ \ \forall x(x \not\approx c \wedge \exists y \neg P(y) \Rightarrow Q(x)) \end{array} \right\},$$
$$\mathscr{E} = \left\{\, e1.\ \ \Diamond\neg Q(x) \,\right\}, \quad \mathscr{S} = \left\{\, s1.\ \ P(x) \Rightarrow \bigcirc\neg P(x) \,\right\}$$

and apply temporal resolution to this. First, we produce the following e-derived step clause from $s1$:

$$d1.\ \ \exists y P(y) \Rightarrow \bigcirc\exists y \neg P(y).$$

Then, we merge $d1$ and $(x \not\approx c \Rightarrow \bigcirc(x \not\approx c))$ from $\mathrm{aug}_{=}(\mathsf{P})$ to give

$$m1.\ \ \forall x(\exists y P(y) \wedge x \not\approx c \Rightarrow \bigcirc(\exists y \neg P(y) \wedge x \not\approx c)).$$

It can be immediately checked that the following formulae are valid

$$\begin{array}{ll} \forall x((\mathscr{U} \wedge \exists y \neg P(y) \wedge x \not\approx c) \Rightarrow Q(x)) & \text{(see } u2), \\ \forall x((\mathscr{U} \wedge \exists y \neg P(y) \wedge x \not\approx c) \Rightarrow (\exists y P(y) \wedge x \not\approx c)) & \text{(see } u1), \end{array}$$

that is, the loop side conditions are valid for $m1$. We apply the eventuality resolution rule to $e1$ and $m1$ and derive a new universal clause

$$u3.\ \ (\forall y \neg P(y) \vee \forall x(x \approx c))$$

which contradicts clauses $u1$ and $i1$ (consequently, the initial termination rule is applied).

Correctness of the presented calculi is straightforward.

Theorem 1. *The rules of $\mathfrak{I}_c$ and $\mathfrak{I}_e$ preserve satisfiability over constant and expanding domains, respectively.*

Proof Considering models, it follows that the temporal resolution rules preserve satisfiability. Consider, for example, the step resolution rule. Let $\mathscr{A} \Rightarrow \bigcirc\mathscr{B}$ be a merged derived clause and assume that $\mathfrak{M}_0 \models^{\mathfrak{a}} \square(\mathscr{A} \Rightarrow \bigcirc\mathscr{B})$, but for some $i \geq 0$, $\mathfrak{M}_i \not\models^{\mathfrak{a}} \neg\mathscr{A}$. Then $\mathfrak{M}_{i+1} \models^{\mathfrak{a}} \mathscr{B}$ in contradiction with the side condition of the rule. ❑

We formulate now completeness results and prove them in Section 5, which is entirely devoted to this issue.

Theorem 2. *If a guarded monodic temporal problem with equality P is unsatisfiable over constant domains, then there exists a successfully terminating derivation in $\mathfrak{I}_c$ from P.*

Theorem 3. *If a guarded monodic temporal problem with equality P is unsatisfiable over expanding domains, then there exists a successfully terminating derivation in $\mathfrak{I}_e$ from P.*

The calculi are complete in the sense that they provides us with a decision procedure when side conditions checks are decidable and with a semi-decision procedure else.

Corollary 1. *Satisfiability of the guarded monodic temporal fragment with equality is decidable by temporal resolution.*

Proof Since there are only finitely many different merged clauses, there are only finitely many different conclusions by the rules of temporal resolution. Now it suffices to note that these side conditions are expressed by first-order guarded formulae with equality (mind our "extended" definition of the guarded fragment, Note 1), and the first-order guarded fragment with equality is decidable [1, 8]. ❑

A complete temporal resolution calculus for the monodic temporal fragment *without equality* for the constant domain case has been presented in [3]. The expanding domain case has been announced in [11] and proved in a technical report [4]. We show that the calculi $\mathfrak{I}_c$ and $\mathfrak{I}_e$, that slightly differ from the calculi used in [3] and [4], are complete for these cases. We briefly discuss the difference between the calculi in Section 5.3.

Theorem 4. *Let an arbitrary monodic temporal problem* without equality *P be unsatisfiable over constant domains. Then there exists a successfully terminating derivation in $\mathfrak{I}_c$ from P.*

Theorem 5. *Let an arbitrary monodic temporal problem* without equality *P be unsatisfiable over expanding domains. Then there exists a successfully terminating derivation in $\mathfrak{I}_e$ from P.*

5 Completeness of Temporal Resolution

The proof of theorems 2 and 3, as well as of theorems 4 and 5, can be obtained by a modification of the corresponding proof of completeness for the constant domain case without equality (see [3], Theorem 2). In short, the proof in [3] proceeds by building a graph associated with a monodic temporal problem, then showing that there is a correspondence between properties of the graph and of the problem, and that all relevant properties are captured by the rules of the proof system. Therefore, if the problem is unsatisfiable, eventually our rules will discover it.

The outlined proof relies on the theorem on existence of a model (see [3], Theorem 3). In Section 5.1 we prove the theorem on existence of a model, Theorem 6, for the constant domain guarded monodic fragment with equality; in Section 5.2 we refine this reasoning for the expanding domain case; and in Section 5.3 we show that the proofs of sections 5.1 and 5.2 can be transfered to arbitrary monodic fragments without equality. It can be seen that the proof of completeness given in [3] holds for all these cases of the theorem on existence of a model considered in sections 5.1–5.3.

5.1 Guarded monodic fragment with equality over constant domains

Let $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$ be a guarded monodic temporal problem with equality. Let $\{P_1(x), \ldots, P_N(x)\}$ and $\{p_1, \ldots, p_n\}$, $N, n \geq 0$, be the sets of all (monadic) predicates (including "predicates" of the form $x \approx c$ for every constant $c \in \mathrm{const}(\mathsf{P})$) and all propositions, respectively, occurring in $\mathscr{S} \cup \mathscr{E} \cup \mathrm{aug}_{=}(\mathsf{P})$.

A *predicate colour* γ is a set of unary literals such that for every $P_i(x) \in \{P_1(x), \ldots, P_N(x)\}$, either $P_i(x)$ or $\neg P_i(x)$ belongs to γ. A predicate colour is called *constant* if $x \approx c \in \gamma$ for some $c \in \mathrm{const}(\mathsf{P})$. A *propositional colour* θ is a sequence of propositional literals such that for every $p_i \in \{p_1, \ldots, p_n\}$, either p_i or $\neg p_i$ belongs to θ. Let Γ be a set of predicate colours. A couple (Γ, θ) is called a *colour scheme* for P. Since P only determines the signature, we may omit P when speaking of colour schemes.

For every colour scheme $\mathscr{C} = \langle \Gamma, \theta \rangle$ let us construct the formulae $\mathscr{F}_\mathscr{C}$, $\mathscr{A}_\mathscr{C}$, $\mathscr{B}_\mathscr{C}$ in the following way. For every $\gamma \in \Gamma$ and for every θ, introduce the conjunctions:

$$F_\gamma(x) = \bigwedge_{L(x) \in \gamma} L(x); \qquad F_\theta = \bigwedge_{l \in \theta} l.$$

Let

$$\begin{gathered} A_\gamma(x) = \bigwedge \{L(x) \mid L(x) \Rightarrow \bigcirc M(x) \in \mathscr{S} \cup \mathrm{aug}_{=}(\mathsf{P}),\ L(x) \in \gamma\}, \\ B_\gamma(x) = \bigwedge \{M(x) \mid L(x) \Rightarrow \bigcirc M(x) \in \mathscr{S} \cup \mathrm{aug}_{=}(\mathsf{P}),\ L(x) \in \gamma\}, \\ A_\theta = \bigwedge \{l \mid l \Rightarrow \bigcirc m \in \mathscr{S},\ l \in \theta\},\ \ B_\theta = \bigwedge \{m \mid l \Rightarrow \bigcirc m \in \mathscr{S},\ l \in \theta\}. \end{gathered}$$

Now $\mathscr{A}_\mathscr{C}$, $\mathscr{B}_\mathscr{C}$, and $\mathscr{F}_\mathscr{C}$ are of the following forms:

$$\begin{gathered} \mathscr{A}_\mathscr{C} = \bigwedge_{\gamma \in \Gamma} \exists x A_\gamma(x) \wedge A_\theta \wedge \forall x \bigvee_{\gamma \in \Gamma} A_\gamma(x),\ \ \mathscr{B}_\mathscr{C} = \bigwedge_{\gamma \in \Gamma} \exists x B_\gamma(x) \wedge B_\theta \wedge \forall x \bigvee_{\gamma \in \Gamma} B_\gamma(x), \\ \mathscr{F}_\mathscr{C} = \bigwedge_{\gamma \in \Gamma} \exists x F_\gamma(x) \wedge F_\theta \wedge \forall x \bigvee_{\gamma \in \Gamma} F_\gamma(x). \end{gathered}$$

We can consider the formula $\mathscr{F}_{\mathscr{C}}$ as a "categorical" formula specification of the quotient structure given by a colour scheme. In turn, the formula $\mathscr{A}_{\mathscr{C}}$ represents the part of this specification which is "responsible" just for "transferring" requirements from the current world (quotient structure) to its immediate successors, and $\mathscr{B}_{\mathscr{C}}$ represents the result of this transferal.

Definition 9 (Canonical merged derived step clauses). *Let* P *be a first-order temporal problem, and* $\mathscr{C}$ *be a colour scheme for* P*. Then the clause*

$$(\mathscr{A}_{\mathscr{C}} \Rightarrow \bigcirc \mathscr{B}_{\mathscr{C}}),$$

is called a canonical merged derived step clause *for* P *(including the* degenerate *clause* **true** $\Rightarrow \bigcirc$**true***). If a conjunction* $A_\gamma(x)$, $\gamma \in \Gamma$*, is empty, that is its truth value is* **true***, then the formula* $\forall x \bigvee_{\gamma\in\Gamma} A_\gamma(x)$ *(or* $\forall x \bigvee_{\gamma\in\Gamma} B_\gamma(x)$*) disappears from* $\mathscr{A}_{\mathscr{C}}$ *(or from* $\mathscr{B}_{\mathscr{C}}$ *respectively). In the propositional case, the clause* $(\mathscr{A}_{\mathscr{C}} \Rightarrow \bigcirc \mathscr{B}_{\mathscr{C}})$ *reduces to* $(A_\theta \Rightarrow \bigcirc B_\theta)$*.*

Definition 10 (Canonical merged step clause). *Let* $\mathscr{C}$ *be a colour scheme,* $\mathscr{A}_{\mathscr{C}} \Rightarrow \bigcirc \mathscr{B}_{\mathscr{C}}$ *be a canonical merged derived step clause, and* $\gamma \in \mathscr{C}$*.*

$$\forall x(\mathscr{A}_{\mathscr{C}} \wedge A_\gamma(x) \Rightarrow \bigcirc(\mathscr{B}_{\mathscr{C}} \wedge B_\gamma(x)))$$

is called a canonical merged step clause*. If the truth value of the conjunctions* $A_\gamma(x)$, $B_\gamma(x)$ *is* **true***, then the canonical merged step clause is just a canonical merged derived step clause. Here,* $\gamma \in \mathscr{C}$ *abbreviates* $\gamma \in \Gamma$*, where* $\mathscr{C} = (\Gamma, \theta)$*.*

Now, given a temporal problem $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$ we define a finite directed graph G as follows. Every vertex of G is a colour scheme $\mathscr{C}$ for P such that $\mathscr{U} \cup \mathscr{F}_{\mathscr{C}}$ is satisfiable. For each vertex $\mathscr{C} = (\Gamma, \theta)$, there is an edge in G to $\mathscr{C}' = (\Gamma', \theta')$, if $\mathscr{U} \wedge \mathscr{F}_{\mathscr{C}'} \wedge \mathscr{B}_{\mathscr{C}}$ is satisfiable. They are the only edges originating from $\mathscr{C}$. A vertex $\mathscr{C}$ is designated as an *initial* vertex of G if $\mathscr{I} \wedge \mathscr{U} \wedge \mathscr{F}_{\mathscr{C}}$ is satisfiable. The *behaviour graph* H of P is the subgraph of G induced by the set of all vertices reachable from the initial vertices.

Definition 11 (Path; Path Segment). *A* path*,* π*, through a behaviour graph,* H*, is a function from* $\mathbb{N}$ *to the vertices of the graph such that for any* $i \geq 0$ *there is an edge* $\langle \pi(i), \pi(i+1) \rangle$ *in* H*. In the similar way, we define a* path segment *as a function from* $[m, n]$*,* $m < n$*, to the vertices of* H *with the same property.*

Lemma 1. *Let* $\mathsf{P}_1 = \langle \mathscr{U}_1, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$ *and* $\mathsf{P}_2 = \langle \mathscr{U}_2, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$ *be two problems such that* $\mathscr{U}_1 \subseteq \mathscr{U}_2$*. Then the behaviour graph of* P_2 *is a subgraph of the behaviour graph of* P_1*.*

Definition 12 (Suitability). *For* $\mathscr{C} = (\Gamma, \theta)$ *and* $\mathscr{C}' = (\Gamma', \theta')$*, let* $(\mathscr{C}, \mathscr{C}')$ *be an ordered pair of colour schemes for a temporal problem* P*. An ordered pair of predicate colours* (γ, γ') *where* $\gamma \in \Gamma$*,* $\gamma' \in \Gamma'$ *is called* suitable *if the formula* $\mathscr{U} \wedge \exists x(F_{\gamma'}(x) \wedge B_\gamma(x))$ *is satisfiable. Similarly, an ordered pair of propositional colours* (θ, θ') *is suitable if* $\mathscr{U} \wedge F_{\theta'} \wedge B_\theta$ *is satisfiable.*

Note that the satisfiability of $\exists x(F_{\gamma'}(x) \wedge B_\gamma(x))$ implies $\models \forall x(F_{\gamma'}(x) \Rightarrow B_\gamma(x))$ as the conjunction $F_{\gamma'}(x)$ contains a valuation at x of *all* predicates occurring in $B_\gamma(x)$.

Note 3. If an ordered pair (γ,γ') is suitable then for every constant $c \in \mathrm{const}(\mathsf{P})$ we have $x \approx c \in \gamma$ iff $x \approx c \in \gamma'$. It implies that if $x \approx c \in \gamma$, then there exist not more than one γ' and not more than one γ'' such that the pairs (γ,γ') and (γ'',γ) are suitable.

Lemma 2. *Let H be the behaviour graph for the problem $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$ with an edge from a vertex $\mathscr{C} = (\Gamma, \theta)$ to a vertex $\mathscr{C}' = (\Gamma', \theta')$. Then*

1. *for every $\gamma \in \Gamma$ there exists a $\gamma' \in \Gamma'$ such that the pair (γ,γ') is suitable;*
2. *for every $\gamma' \in \Gamma'$ there exists a $\gamma \in \Gamma$ such that the pair (γ,γ') is suitable;*
3. *the pair of propositional colours (θ,θ') is suitable;*

Definition 13 (Run/E-Run). *Let π be a path through a behaviour graph H of a temporal problem P, and $\pi(i) = (\Gamma_i, \theta_i)$. By a* run *in π we mean a function $r(n)$ from $\mathbb{N}$ to $\bigcup_{i\in\mathbb{N}} \Gamma_i$ such that for every $n \in \mathbb{N}$, $r(n) \in \Gamma_n$ and the pair $(r(n), r(n+1))$ is suitable. In a similar way, we define a* run segment *as a function from $[m,n]$, $m < n$, to $\bigcup_{i\in\mathbb{N}} \Gamma_i$ with the same property.*

A run r is called an e-run *if for all $i \geq 0$ and for every non-ground eventuality $\Diamond L(x) \in \mathscr{E}$ there exists $j > i$ such that $L(x) \in r(j)$.*

Let π be a path, the set of all runs in π is denoted by $\mathscr{R}(\pi)$, and the set of all e-runs in π is denoted by $\mathscr{R}_e(\pi)$. If π is clear, we may omit it.

A run r is called a *constant run* if $x \approx c \in r(i)$ for some $i \geq 0$. Note that if a run is constant and $x \approx c \in r(i)$ for some $i \geq 0$, then $x \approx c \in r(j)$ for all $j \in \mathbb{N}$. If, for two runs r and r', a constant c, and some $i \geq 0$ we have $x \approx c \in r(i)$ and $x \approx c \in r'(i)$, then $r = r'$.

Let $\rho_{\mathscr{C}}$ be a mapping from $\mathrm{const}(\mathsf{P})$ to Γ such that $x \approx c \in \rho_{\mathscr{C}}(c)$. Then the function defined as $r_c(n) = \rho_{\mathscr{C}_n}(c)$ is the unique constant run "containing" c.

Theorem 6 (Existence of a model). *Let $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$ be a temporal problem. Let H be the behaviour graph of P. If both the set of initial vertices of H is non-empty and the following conditions hold*

1. *For every vertex $\mathscr{C} = (\Gamma, \theta)$, predicate colour $\gamma \in \Gamma$, and non-ground eventuality $\Diamond L(x) \in \mathscr{E}$, there exist a vertex $\mathscr{C}' = (\Gamma, \theta)$ and a predicate colour $\gamma' \in \Gamma'$ such that $((\mathscr{C},\gamma) \to^+ (\mathscr{C}',\gamma') \wedge L(x) \in \gamma')$;*
2. *For every vertex $\mathscr{C} = (\Gamma, \theta)$ and ground eventuality $\Diamond l \in \mathscr{E}$ there exist a vertex $\mathscr{C}' = (\Gamma, \theta)$ such that $(\mathscr{C} \to^+ \mathscr{C}' \wedge l \in \theta')$;*

then P has a model.[6] *Here $(\mathscr{C},\gamma) \to^+ (\mathscr{C}',\gamma')$ denotes that there exists a path π from $\mathscr{C}$ to $\mathscr{C}'$ such that γ and γ' belong to a run in π; $\mathscr{C} \to^+ \mathscr{C}'$ denotes that there exists a path from $\mathscr{C}$ to $\mathscr{C}'$.*

[6] Following [3], in the original version of this paper, Theorem 6 contained one more condition: for every vertex $\mathscr{C} = (\Gamma, \theta)$, non-ground eventuality $L(x) \in \mathscr{E}$, and constant $c \in \mathrm{const}(\mathsf{P})$ there exists a vertex $\mathscr{C}' = (\Gamma, \theta)$ such that $\left(\mathscr{C} \to^+ \mathscr{C}' \wedge L(x) \in \rho_{\mathscr{C}'}(c)\right)$. This condition was essential for the completeness of the calculus without equality presented in [3], and it led to the introduction of so called *constant flooding*, see [3]. However, one of the referees noticed that, under definitions of this paper (after including equality into consideration), condition (1) already implies the additional condition leading to the obsolescence of constant flooding.

Proof The proof relies on the following lemma, whose proof was given in [3].

Lemma 3. *Under the conditions of Theorem 6, there exists a path π through H where:*

(a) $\pi(0)$ *is an initial vertex of* H*;*
(b) *for every colour scheme* $\mathscr{C} = \pi(i)$, $i \geq 0$, *and every ground eventuality literal* $\Diamond l \in \mathscr{E}$ *there exists a colour scheme* $\mathscr{C}' = \pi(j)$, $j > i$, *such that* $l \in \theta'$*;*
(c) *for every colour scheme* $\mathscr{C} = \pi(i)$, $i \geq 0$, *and every predicate colour* γ *from the colour scheme there exists an e-run* $r \in \mathscr{R}_e(\pi)$ *such that* $r(i) = \gamma$*; and*
(d) *for every constant* $c \in \mathrm{const}(P)$, *the function* $r_c(n)$ *defined by* $r_c(n) = \rho_{\mathscr{C}_n}(c)$ *is an e-run in* π.

Let $\pi = \mathscr{C}_0, \ldots, \mathscr{C}_n, \ldots$ be a path through H defined by Lemma 3. Let $\mathscr{G}_0 = \mathscr{I} \cup \{\mathscr{F}_{\mathscr{C}_o}\}$ and $\mathscr{G}_n = \mathscr{F}_{\mathscr{C}_n} \wedge \mathscr{B}_{\mathscr{C}_{n-1}}$ for $n \geq 1$. According to the definition of a behaviour graph, the set $\mathscr{U} \cup \{\mathscr{G}_n\}$ is satisfiable for every $n \geq 0$.

Now, Lemma 8 from [9], that captures properties of the guarded fragment, can be reformulated as follows.

Lemma 4. *Let* κ *be a cardinal,* $\kappa \geq \aleph_0$. *For every* $n \geq 0$, *there exists a model* $\mathfrak{M}_n = \langle D, I_n \rangle$ *of* $\mathscr{U} \cup \{\mathscr{G}_n\}$ *such that for every* $\gamma \in \Gamma_n$ *the set* $\{a \in D \mid \mathfrak{M}_n \models F_\gamma(a)\}$ *is of cardinality* 1 *if* γ *is a constant colour and of cardinality* κ *otherwise.*

Following [10, 2, 9] take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set $\mathscr{R}_e$. Let r be a run in $\mathscr{R}_e$. We define the set D_r as $\{\langle r, 0\rangle\}$ if r is a constant run and as $\{\langle r, \xi\rangle \mid \xi < \kappa\}$ otherwise.

Let us define a domain $D = \bigcup\limits_{r \in \mathscr{R}_e} D_r$. For every $n \in \mathbb{N}$ we have $D = \bigcup\limits_{\gamma \in \Gamma_n} D_{(n,\gamma)}$, where $D_{(n,\gamma)} = \{\langle r, \xi\rangle \in D \mid r(n) = \gamma\} = \bigcup\limits_{r \in \mathscr{R}_e,\, r(n)=\gamma} D_r$. Then $\left|D_{(n,\gamma)}\right| = 1$ if γ is a constant colour and $\left|D_{(n,\gamma)}\right| = \kappa$ otherwise.

Hence, by Lemma 4, for every $n \in \mathbb{N}$ there exists a structure $\mathfrak{M}_n = \langle D, I_n\rangle$ satisfying $\mathscr{U} \cup \{\mathscr{G}_n\}$ such that $D_{(n,\gamma)} = \{\langle r, \xi\rangle \in D \mid \mathfrak{M}_n \models F_\gamma(\langle r, \xi\rangle)\}$. Moreover, $c^{I_n} = \langle r_c, 0\rangle$ for every constant $c \in \mathrm{const}(\mathsf{P})$. A potential first order temporal model is $\mathfrak{M} = \langle D, I\rangle$, where $I(n) = I_n$ for all $n \in \mathbb{N}$. To be convinced of this we have to check validity of the step and eventuality clauses. (Recall that satisfiability of $\mathscr{I}$ in $\mathfrak{M}_0$ is implied by satisfiability of $\mathscr{G}_0$ in $\mathfrak{M}_0$.)

Let $\Box\forall x(L_i(x) \Rightarrow \bigcirc M_i(x))$ be an arbitrary step clause; we show that it is true in $\mathfrak{M}$. Namely, we show that for every $n \geq 0$ and every $\langle r, \xi\rangle \in D$, if $\mathfrak{M}_n \models L_i(\langle r, \xi\rangle)$ then $\mathfrak{M}_{n+1} \models M_i(\langle r, \xi\rangle)$. Suppose $r(n) = \gamma \in \Gamma_n$ and $r(n+1) = \gamma' \in \Gamma'$, where (γ, γ') is a suitable pair in accordance with the definition of a run. It follows that $\langle r, \xi\rangle \in D_{(n,\gamma)}$ and $\langle r, \xi\rangle \in D_{(n+1,\gamma')}$, in other words $\mathfrak{M}_n \models F_\gamma(\langle r, \xi\rangle)$ and $\mathfrak{M}_{n+1} \models F_{\gamma'}(\langle r, \xi\rangle)$. Since $\mathfrak{M}_n \models L_i(\langle r, \xi\rangle)$ then $L_i(x) \in \gamma$. It follows that $M_i(x)$ is a conjunctive member of $B_\gamma(x)$. Since the pair (γ, γ') is suitable, it follows that the conjunction $\exists x(F_{\gamma'}(x) \wedge B_\gamma(x))$ is satisfiable and, moreover, $\models \forall x(F_{\gamma'}(x) \Rightarrow B_\gamma(x))$. Together with $\mathfrak{M}_{n+1} \models F_{\gamma'}(\langle r, \xi\rangle)$ this implies that $\mathfrak{M}_{n+1} \models M_i(\langle r, \xi\rangle)$.

Let $(\Box\forall x)\Diamond L(x)$ be an arbitrary eventuality clause. We show that for every $n \geq 0$ and every $\langle r, \xi\rangle \in D$, $r \in \mathscr{R}_e, \xi < \kappa$, there exists $m > n$ such that $\mathfrak{M}_m \models L(\langle r, \xi\rangle)$. Since r is an e-run, there exists $\mathscr{C}' = \pi(m)$ for some $m > n$ such that $r(m) = \gamma' \in \Gamma'$ and $L(x) \in \gamma'$. It follows that $\langle r, \xi \rangle \in D_{(m,\gamma')}$, that is $\mathfrak{M}_m \models F_{\gamma'}(\langle r, \xi \rangle)$. In particular, $\mathfrak{M}_m \models L(\langle r, \xi \rangle)$. Propositional step and eventuality clauses are treated in a similar way. ❑

5.2 Guarded monodic fragment with equality over expanding domains

We here outline how to modify the proof of Theorem 6 for the case of expanding domains. All the definitions and properties from the previous section are transfered here with the following exceptions.

Now, the universally quantified part does not contribute either to $\mathscr{A}$ or $\mathscr{B}$:

$$\mathscr{A}_{\mathscr{C}} = \bigwedge_{\gamma \in \Gamma} \exists x A_\gamma(x) \wedge A_\theta, \qquad \mathscr{B}_{\mathscr{C}} = \bigwedge_{\gamma \in \Gamma} \exists x B_\gamma(x) \wedge B_\theta.$$

This change affects the suitability of predicate colours.

Lemma 5 (analog of Lemma 2). *Let H be the behaviour graph for the problem $\mathsf{P} = \langle \mathscr{U}, \mathscr{I}, \mathscr{S}, \mathscr{E} \rangle$ with an edge from a vertex $\mathscr{C} = (\Gamma, \theta)$ to a vertex $\mathscr{C}' = (\Gamma', \theta')$. Then*

1. *for every $\gamma \in \Gamma$ there exists a $\gamma' \in \Gamma'$ such that the pair (γ, γ') is suitable;*
3. *the pair of propositional colours (θ, θ') is suitable;*

Note that the missing condition (2) of Lemma 2 does not hold in the expanding domain case. However, under the conditions of Lemma 5, if $\gamma' \in \Gamma'$ contains $x \approx c$, there always exists a $\gamma \in \Gamma$ such that the pair (γ, γ') is suitable.

Since for a non-constant predicate colour γ there may not exist a colour γ' such that the pair (γ', γ) is suitable, the notion of a run is reformulated.

Definition 14 (Non-constant run). *Let π be a path through a behaviour graph H of a temporal problem P. By a* non-constant run *in π we mean a function $r(n)$ mapping its domain, $dom(r) = \{n \in \mathbb{N} \mid n \geq n_0\}$ for some $n_0 \in \mathbb{N}$, to $\bigcup_{i \in \mathbb{N}} \Gamma_i$ such that for every $n \in dom(r)$, $r(n) \in \Gamma_n$, $r(n)$ is not a constant predicate colour, and the pair $(r(n), r(n+1))$ is suitable. (Constant runs are defined as in the constant domain case.)*

5.3 Monodic fragment without equality

Note that the only place where the proof of Theorem 6, given in Section 5.1, and its counterpart for the expanding domain case, given in Section 5.2, need the problem to be guarded is Lemma 4. If a monodic temporal problem P *does not* contain equality, Lemma 4 holds regardless the problem being guarded or not.

Consider the constant domain case (similar reasoning takes place for the expanding domain case). Let $\mathscr{U} \cup \{\mathscr{G}_n\}$ be satisfiable, and let $\mathfrak{M}_n$ be its model. Let $\mathscr{C}_n = (\Gamma_n, \theta_n)$. For a constant $c \in \mathrm{const}(\mathsf{P})$, let us define Γ_c to be $\{\gamma \in \Gamma_n \mid x \approx c \in \gamma\}$; the set Γ_c is a singleton. Let Γ_n' be obtained by eliminating all equations and disequations from Γ_n. Let us define now the formula $\mathscr{F}'_{\mathscr{C}_n}$ as

$$\bigwedge_{\gamma \in \Gamma_n'} \exists x F_\gamma(x) \wedge \bigwedge_{c \in \mathrm{const}(\mathsf{P}),\, \gamma \in \Gamma_c} F_\gamma(c) \wedge F_{\theta_n} \wedge \forall x \bigvee_{\gamma \in \Gamma_n'} F_\gamma(x).$$

Analogously, we define the formulae $\mathscr{B}'_{\mathscr{C}_n}$ and $\mathscr{G}'_n = \mathscr{F}'_{\mathscr{C}_n} \wedge \mathscr{B}'_{\mathscr{C}_{n-1}}$. It is not hard to see that since $\mathscr{U} \cup \{\mathscr{G}_n\}$ is satisfiable, $\mathscr{U} \cup \{\mathscr{G}'_n\}$ is satisfiable. As $\mathscr{U} \cup \{\mathscr{G}'_n\}$ does not contain equality, from classical model theory, there exists a model $\mathfrak{M}'_n = \langle D', I'_n \rangle$ of $\mathscr{U} \cup \{\mathscr{G}'_n\}$ such that for every $\gamma \in \Gamma'_n$ the set $D'_{(n,\gamma')} = \{a \in D' \mid \mathfrak{M}'_n \models F_\gamma(a)\}$ is of cardinality κ, and for all $c_1, c_2 \in \mathrm{const}(\mathsf{P})$, $I'_n(c_1) = I'_n(c_2)$ iff $I_n(c_1) = I_n(c_2)$. Note that $\mathfrak{M}_n$ is a model for $\mathscr{U} \cup \{\mathscr{G}_n\}$. Obviously, a constant predicate colour γ is true on a single element of the domain D; disequations such as $x \not\approx c$ exclude only finitely many elements.

As already mentioned in Section 4, Note 2, instead of extending P with step clauses of the form (1) and (2), we could consider derived step clauses of the form (6). Completeness of the resulting calculus for the constant domain case has been presented in [3]. Completeness for the expanding domain case can be obtained by combining the proof technique from [3] with the previous section.

6 Fine-grained temporal superposition

The main drawback of the calculi introduced in Section 4 is that the notion of a merged step clause is quite involved and the search for appropriate merging of simpler clauses is computationally hard. Finding *sets* of such full merged step clauses needed for the temporal resolution rule is even more difficult.

This problem has been tackled for the expanding domain case without equality in [11]. The expanding domain case is simpler firstly because merged e-derived step clauses are simpler (formulae of the form (5) do not contribute to them) and, secondly, because conclusions of all inference rules of $\mathfrak{I}_e$ are first-order clauses. We have introduced in [11] a calculus where the inference rules of $\mathfrak{I}_e$ were refined into smaller steps, more suitable for effective implementation. We have also shown that the search for premises for the eventuality resolution rule can be implemented by means of a search algorithm based on step resolution. We called the resulting calculus *fine-grained resolution*.

In the same way as we have used first-order resolution to obtain a complete fine-grained resolution calculus for the expanding domain monodic fragment without equality, we can use first-order superposition to obtain a *fine-grained superposition* calculus for the expanding domain *guarded* monodic fragment *with equality*. In order to do that, we apply ideas from [11] to a first-order superposition decision procedure for the guarded fragment with equality given in [7]. Fine-grained superposition takes as input an augmented temporal problem transformed in clausal form: the universal and initial parts are clausified, as if there is no connection with temporal logic at all.

In contrast to $\mathfrak{I}_e$ which generates only universal formulae, fine-grained superposition might generate initial, universal, or step clauses of the form $C \Rightarrow \bigcirc D$, where C is a *conjunction* of propositional literals and unary literals of the form $L(x)$, $x \approx c$, or $x \not\approx c$; and ground formulae of the form $L(c)$, where $L(x)$, is a unary literal and c is a constant occurring in the originally given problem; D is a *disjunction* of arbitrary literals.

Following [11], we allow only the right-hand side of step clauses to be involved in an inference rule and impose a restriction on mgus. For example, the *step paramodulation* rule will take the following form:

$$\frac{C_1 \Rightarrow \bigcirc(D_1 \vee L[s]) \quad C_2 \Rightarrow \bigcirc(D_2 \vee t \approx u)}{(C_1 \wedge C_2)\sigma \Rightarrow \bigcirc(D_1 \vee D_2 \vee L[u])\sigma} \quad \text{and} \quad \frac{C_1 \Rightarrow \bigcirc(D_1 \vee L[s]) \quad D_2 \vee t \approx u}{C_1\sigma \Rightarrow \bigcirc(D_1 \vee D_2 \vee L[u])\sigma},$$

where $C_1 \Rightarrow \bigcirc(D_1 \vee L[s])$ and $C_2 \Rightarrow \bigcirc(D_2 \vee t \approx u)$ are step clauses, $D_2 \vee t \approx u$ is a universal clause, σ is an mgu of s and t *such that* σ *does not map variables from* C_1 *or* C_2 *(or just from* C_1*) into a Skolem constant or a Skolem functional term*. This restriction justifies skolemisation: Skolem constants and functions do not "sneak" in the left-hand side of step clauses, and, hence, Skolem constants from different moments of time do not interact.

Other rules of fine-grained superposition can be obtained in a similar way from the rules of the calculus given in [7]. Correctness and completeness of the resulting calculus for the expanding domain guarded monodic fragment with equality can be proved just as the corresponding properties of fine-grained resolution has been proved in [11].

Example 3. Consider a guarded monodic temporal problem, P, unsatisfiable over expanding domains:

$$\mathscr{I} = \{i1.\ c \not\approx d\}, \quad \mathscr{U} = \{u1.\ \forall x(\neg P(x) \vee x \approx c)\}$$
$$\mathscr{S} = \{s1.\ \mathbf{true} \Rightarrow \bigcirc P(d)\}, \quad \mathscr{E} = \emptyset.$$

Although this problem is not in DSNF, it can be easily reduced to DSNF by renaming; however, such a reduction would complicate understanding.

First, we give a "course-grained" refutation. The right-hand side of a merged e-derived step clause

$$m1.\ \exists x(x \approx d \wedge x \not\approx c) \Rightarrow \bigcirc \exists x(x \approx d \wedge x \not\approx c \wedge P(d))$$

contradicts to the universal part, and, by the step resolution rule, we conclude $\forall x(x \not\approx d \vee x \approx c)$ which contradicts the initial part.

We show now how fine-grained superposition helps us to find the required merged e-derived step clause $m1$. We need the following step clauses from $\mathrm{aug}_{=}(\mathsf{P})$:

$$a1.\ y \not\approx d \Rightarrow \bigcirc y \not\approx d \quad \text{and} \quad a2.\ x \approx c \Rightarrow \bigcirc x \approx c.$$

We now derive:

$s2.$ **true**	$\Rightarrow \bigcirc d \approx c$	(resolution $u1$ and $s1$)
$s3.\ y \not\approx d$	$\Rightarrow \bigcirc y \not\approx c$	(paramodulation $s2$ and $a1$)
$s4.\ y \not\approx d \wedge x \approx c$	$\Rightarrow \bigcirc x \not\approx y$	(paramodulation $s3$ and $a2$)
$s5.\ x \not\approx d \wedge x \approx c$	$\Rightarrow \bigcirc$**false**	(reflexivity resolution $s4$)

We convert the step clause $s5$ into the universal clause $u2.\ x \approx d \vee x \not\approx c$ and resolve with $i1$ giving $i2.\ c \not\approx c$. Finally, we derive an empty clause by reflexivity resolution.

7 Concluding remarks

In this paper we have considered the basis for mechanising the extension of monodic FOTL by equality. In particular, we have presented resolution calculi for the guarded monodic fragment with equality over both constant and expanding domains. Provided that there exists a first-order decision procedure for side conditions of all inference rules, then these calculi provide the basis for decision procedures. As indicated in section 6, a more practical approach is being developed (for the expanding domain case) based on fine-grained superposition for the guarded monodic fragment. Extension and implementation of this approach represents much of our future work. Finally, we acknowledge support from EPSRC via research grant GR/L87491 and thank the (anonymous) referees of the LPAR conference for their helpful and insightful comments.

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